

Influence of Rotation on the Amount of Phantom Matter around wormholes

Alexander Shatskiy¹

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Abstract

Static (spherically symmetrical) and stationary solutions for wormholes are considered. The visibility horizon, which characterizes the differences between black holes and passing wormholes, is determined in an invariant way. It is shown that the rotation of wormholes does not affect the amount of phantom matter that surrounds them.

1 INTRODUCTION

In recent work in relativistic astrophysics, interest in solutions with wormholes (WHs) has increased, due, in particular, to the future construction of high accuracy radio interferometers that will be able to distinguish WHs from other objects (such as black holes).

A fundamental and characteristic property of a WH is its throat, through which physical bodies can pass. Spacetime is strongly curved near the throat. This curvature reaches values corresponding to the horizon of a black hole with the same mass.

A necessary condition distinguishing a black hole from a WH is the absence of an event horizon for the latter. Therefore, in Section 3, we find an invariant criterion for the existence of an event horizon in an axially symmetrical system.

These properties of WHs are maintained by the presence of exotic matter surrounding the throat. Due to this exotic matter, the radial pressure near the WH throat is less than the negative of the energy density: $p_{\parallel} < -\varepsilon$ (a super-rigid equation of state). This matter is called phantom matter. We show in Sections 3-5 that rotation of the WH does not influence the required amount of surrounding phantom matter.

2 SPHERICALLY SYMMETRICAL CASE

The solution for WHs in the spherically symmetrical case was studied in detail, for example, in [1], where it was shown that the spherically symmetrical solution for a WH makes a smooth transition to the solution at the horizon² and the WH becomes unpassable.

The case of the small difference of a WH from an electrically (or magnetically) charged Reisner-Nordstrem black hole may be of practical interest [2].

Let us consider matter with a linear equation of state ($p = \text{const} \cdot \varepsilon$), when the coefficients of the energy density are constant and equal in magnitude to

$$w \equiv 1 + \delta = -p_{\parallel}/\varepsilon = p_{\perp}/\varepsilon, \quad x = r/r_h, \quad \xi(x) = \varepsilon(r)/\varepsilon(r_h), \quad y(x) = x \cdot [1 + 1/g_{rr}(x)]. \quad (1)$$

Here, r_h is the radius of the horizon for a black hole with the same total mass as the WH. We now write the metric for the WH, which has, in the general case, the form

$$ds^2 = \exp(2\phi) \cdot [dt^2] - (1 - y(x)/x)^{-1} \cdot [dr^2] - r^2 \cdot ([d\theta^2] + \sin^2 \theta [d\varphi^2]). \quad (2)$$

¹Astro Space Center, Lebedev Physical Institute, Moscow, Russia; shatskiy@asc.rssi.ru

²The definition of the horizon for the more general case will be given in Section 3.

The equations for this metric can be written in the convenient form [1]:

$$\begin{aligned} y(x) &= 2 - \int_x^\infty \xi x^2 dx / w \\ \frac{1}{\xi} \frac{d\xi}{dx} &= -4/x + \frac{\delta}{2w} \cdot (\xi x - y/x^2)/(1 - y/x) \\ \exp[\phi(x)] &= \text{const}/(\xi x^4)^{w/\delta} \end{aligned} \quad (3)$$

When $\delta = 0$, this solution makes a transition to the solution of Reisner-Nordstrom solution containing the horizon: $y_{[\delta=0]}(x) = 2 - 1/x$, $\xi_{[\delta=0]}(x) = 1/x^4$, $\exp[\phi_{[\delta=0]}(x)] = 1 - 1/x$. We obtain for the solution (3) an analytical expression representing a first approximation in the small correction δ . We denote $z \equiv (1 - 1/x)$. In a linear approximation in δ , Eq. (3) for ξ can be re-written in the form

$$\frac{\partial \ln(\xi x^4)}{\partial x} = -\delta \cdot \frac{\partial \ln(z)}{\partial x} \quad (4)$$

Taking into account the requirement that ξ coincide with $\xi_{[\delta=0]}$ at infinity, we obtain using this relation

$$\xi x^4 = 1/z^\delta, \quad y(x) = 1 + z^{1-\delta}, \quad \exp(\phi) = z^{1+\delta}. \quad (5)$$

The radius of the throat $r_0 = r_h \cdot x_0$ is determined by the transcendental equation $y(x_0) = x_0$, from which we can obtain

$$\delta = -\ln(x_0)/\ln(1 - 1/x_0). \quad (6)$$

The parameter δ is determined by the equation of state of the matter in the WH, including quantum corrections: $\delta \equiv w - 1 > 0$. For example, when $x_0 = 1.001$, $\delta \approx 1.4 \cdot 10^{-4}$.

3 ROTATING WORMHOLES

In the simplest case, the metric tensor for a rotating WH can be obtained by adding a non-diagonal term to the spherically symmetrical WH metric. It is convenient to replace the dr^2 term in (2) by introducing a new variable l that is equal to zero at the throat:

$$dl^2 \equiv (1 - y(x)/x)^{-1} dr^2 \quad (7)$$

The metric then takes the form

$$ds^2 = g_{tt}(l) \cdot [dt^2] - [dl^2] - r^2(l) \cdot \left([d\theta^2] + \sin^2 \theta [d\varphi^2] \right) + 2f(l) \cdot \sin^2 \theta \cdot [dt \cdot d\varphi]. \quad (8)$$

The WH metric can be written in the form (8) only in the case of slow rotation. Otherwise, a θ dependence will appear in the metric coefficients, apart from the fact that this dependence also appears in $g_{t\varphi}$ and $g_{\varphi\varphi}$ (see [6],[7]).

We introduce the required notation for the cylindrical coordinate ρ and the modulus of the metric tensor g :

$$\rho \equiv r(l) \sin \theta, \quad g \equiv g_{tt} r^2 \rho^2 + r^2 g_{t\varphi}^2. \quad (9)$$

A prime will denote a derivative with respect to l .

A necessary condition for the existence of a WH is the absence of an event horizon. The physical meaning of the event horizon is defined in [3].

The mathematical definition of the event horizon is the zero hypersurface having the property that it transmits the world lines of moving particles in only one direction [4]. Thus, a length element ds should be equal to zero in the direction normal to this hypersurface. Precisely this property implies that world lines of particles or light rays (directed into the future) can cross this hypersurface in only one direction.

We now introduce the concept of the invariant velocity V , which coincides with the usual three-dimensional velocity for moving particles in the non-relativistic case, and is equal to unity at the event horizon, independent of the chosen reference frame.

Thus, one definition for V that satisfies the above properties is the given by the expression

$$V^2 \equiv 1 - \frac{ds^2}{dt^2} = 1 - \frac{1}{(U^t)^2}, \quad (10)$$

where U^i are the components of the 4-velocity of the particle. Note that $V^2 \rightarrow v^2$ in the non-relativistic limit, where v is the usual non-relativistic velocity.

To express the invariant velocity in terms of the metric components, we use the fact that there exist four integrals of motion for particles in an axially symmetrical field [2]. Two of these integrals (for an uncharged particle) are U_t and U_φ .

$U_t = E_0/m$ expresses the conservation of energy and $U_\varphi = L/m$ the conservation of angular momentum for the particle.

Hence, (10) can be re-written in the form

$$V^2 = 1 - \left(\frac{m}{E_0}\right)^2 \cdot \frac{1}{(g^{tt} + g^{t\varphi} L/E_0)^2}. \quad (11)$$

Note that this definition of the event horizon is valid for any axially symmetrical metric (including the Newman-Kerr metric [2]).

In the case of a wormhole (see Table 1 in Appendix 8), the horizon is defined as the geometric locus of points for which $g_{tt} + (f \sin \theta/r)^2 = 0$ or $g = 0$. This surface is egg shaped³ and elongated along the axis of rotation.

The spherical surface $g_{tt}(l) = 0$ lies outside the horizon, touching it at the poles, and has the meaning of the outer boundary of the ergosphere.

It follows from the above that the WH throat is located outside the ergosphere, and the quantity $1/r'^2$ is simultaneously for the metric (8) the component $-g_{rr} \rightarrow -\infty$ for $l \rightarrow 0$. Then, particles falling through the throat will not reach the horizon (which should be absent for a WH).

4 THE EINSTEIN EQUATION

In the general case, it is convenient to write the Einstein equation in the form

$$R_{ik} = 8\pi \left(T_{ik} - \frac{1}{2} g_{ik} T \right). \quad (12)$$

We use σ^i to denote the zero 4-vector of a photon with zero angular momentum relative to the center of the system. Since the angular momentum is determined by the covariant component of the 4-velocity, $\sigma_\varphi = 0$ by definition. We can then present σ in the form

$$\sigma_i = F_0 \cdot \{1/\sqrt{g^{tt}}, 1, 0, 0\}; \quad \sigma^i = F_0 \cdot \{\sqrt{g^{tt}}, -1, 0, g^{t\varphi}/\sqrt{g^{tt}}\}; \quad \sigma_i \sigma^i = 0, \quad (13)$$

where F_0 is any function. It is convenient to choose this function to be $F_0 = \sqrt{g_{tt}g^{tt}}$. In this case, $\sigma_i u^i = 1$, where u^i is the 4-velocity of the matter in the comoving reference frame (see Table 21 in Appendix 8).

Performing a scalar multiplication of both sides of (12) by the 4-vector σ^i twice yields

$$R_{ik} \sigma^i \sigma^k = 8\pi T_{ik} \sigma^i \sigma^k \equiv 8\pi \Pi. \quad (14)$$

³The coordinates for this surface can be reduced to a spherical form via an appropriate transform; in this case, the ergosphere will have the form of a flattened ellipsoid.

The scalar Π is directly proportional to the energy density measured by an observer in a reference frame tied to the photon. It stands to reason that this reference frame is meaningless, or rather has meaning only in the asymptotic limit for a system whose velocity approaches the velocity of light.

In the special case of the energy-momentum tensor T_{ik} of an ideal fluid (see Table 21 in Appendix 8), this scalar is equal to $\Pi = \varepsilon + p$.

Its physical meaning for ordinary matter means that Π cannot be negative. This requirement is called the zero-energy condition (NEC), and matter that violates this condition is called phantom, or exotic, matter. To test the violation of the zero-energy condition for a rotating wormhole, we must calculate the necessary components of the Ricci tensor.

5 THE CURVATURE TENSOR

The Ricci tensor has the form

$$R_{ik} = \partial_n(\Gamma_{ik}^n) - \partial_k(\Gamma_{in}^n) + \Gamma_{ik}^n \cdot \Gamma_{nm}^m - \Gamma_{in}^m \cdot \Gamma_{km}^n$$

The required Cristoffel symbols are

$$\Gamma_{kn}^i = \frac{1}{2} g^{im} (\partial_n g_{mk} + \partial_k g_{mn} - \partial_m g_{kn}),$$

and the inverse metric tensor g_{ik} was calculated and presented in Tables 20 and 22 in Appendix 8.

Using these results, we obtain

$$R_{tt} = \frac{g''_{tt}}{2} - \frac{\rho^2(f f' g'_{tt} - f'^2 g_{tt})}{2g} + \frac{f^2 g_{tt} \sin^2(2\theta)}{2g} + \frac{\rho^2 g'_{tt} (g_{tt} (r^2)' - g'_{tt} r^2)}{4g} + \frac{g'_{tt} r'}{2r}, \quad (15)$$

$$R_{ll} = - \left[\frac{\rho^2((r^2 g_{tt})' + 2ff' \sin^2 \theta)}{2g} \right]' - \left[\frac{\rho^2(r^2 g'_{tt} + ff' \sin^2 \theta)}{2g} \right]^2 - \\ - \left[\frac{\rho^2(f f' \sin^2 \theta + 2g_{tt} rr')}{2g} \right]^2 - \frac{r^2 \rho^6 (f/r^2)' (f g'_{tt} - f' g_{tt})}{2g^2} - \frac{r''}{r}, \quad (16)$$

$$R_{t\varphi} = \frac{f'' r^2 \sin^2 \theta + 2f \cos(2\theta) + f' r r' \sin^2 \theta}{2r^2} - \\ - \frac{(f' g'_{tt} r^2 + f f'^2) r^2 \sin^4 \theta + f g_{tt} r^2 \sin^2(2\theta) + (2f' g_{tt} - 4f g'_{tt}) r^3 r' \sin^4 \theta}{4g}, \quad (17)$$

$$R_{\varphi\varphi} = \frac{2\rho^4 r'^2 g_{tt} + 0.5r^4 g_{tt} \sin^2(2\theta) - rr' \rho^4 g'_{tt} - \rho^6 f'(f/r^2)'}{2g} - \\ - \frac{r''}{r} \rho^2 - 2r'^2 \sin^2 \theta - \cos(2\theta). \quad (18)$$

6 PHANTOM MATTER NEAR A WORMHOLE

In a spherically symmetrical WH, phantom matter appears near the throat [5]. Using (13-18), we can find the value of the scalar at the throat (Π_0) for a rotating WH, taking into account that $r' = 0$ at the throat, and any derivative with respect to l can be replaced as follows: $\frac{\partial}{\partial l} \rightarrow r' \frac{\partial}{\partial r}$. Therefore, at the throat, we must include only terms without derivatives and with second derivatives with respect to l .

We thus obtain at the throat (in linear approximation in respect to f^2):

$$\Pi_0 = F_0^2 \left[-\frac{2r''}{r} + \frac{f^2(3 \cos^2 \theta - 1)}{g_{tt} r^4} \right] \quad (19)$$

Using reasoning analogous to the spherically symmetrical case [5], we can convince ourselves that there will also always be a region with phantom matter for a rotating WH. The integral of Π_0 over the all angle $d\Omega = \sin \theta d\theta |_0^\pi d\varphi |_{-\pi}^\pi$ yields the same value for the quantity of phantom matter near the throat as in the absence of rotation. It is possible that this result is due to the smallness of the rotation, and will not be valid for more extreme cases.

7 CONCLUSION and ACKNOWLEDGMENTS

In a linear approximation in the angular velocity, the rotation of a WH does not influence the amount of phantom matter surrounding the WH.

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8 Appendix (tables)

Metric tensor:

g_{ik}	t	l	θ	φ	g^{ik}	t	l	θ	φ
t	g_{tt}	0	0	$f \sin^2 \theta$	t	$r^2 \rho^2/g$	0	0	$f \rho^2/g$
l	0	-1	0	0	l	0	-1	0	0
θ	0	0	$-r^2$	0	θ	0	0	$-1/r^2$	0
φ	$f \sin^2 \theta$	0	0	$-\rho^2$	φ	$f \rho^2/g$	0	0	$-r^2 g_{tt}/g$

(20)

Energy-momentum tensor for an ideal fluid in a frame co-moving with the matter:

$$T_{ik} = (\varepsilon + p)u_i u_k - pg_{ik}; \quad u^i = \{1/\sqrt{g_{tt}}, 0, 0, 0\}; \quad u_i = \{\sqrt{g_{tt}}, 0, 0, g_{t\varphi}/\sqrt{g_{tt}}\}.$$

$T_{ik} - \frac{1}{2}g_{ik}T$	t	l	θ	φ
t	$\frac{1}{2}(\varepsilon + 3p)g_{tt}$	0	0	$\frac{1}{2}(\varepsilon + 3p)g_{t\varphi}$
l	0	$\frac{1}{2}(\varepsilon - p)$	0	0
θ	0	0	$\frac{1}{2}r^2(\varepsilon - p)$	0
φ	$\frac{1}{2}(\varepsilon + 3p)g_{t\varphi}$	0	0	$(\varepsilon + p)g_{t\varphi}^2/g_{tt} + \frac{1}{2}\rho^2(\varepsilon - p)$

(21)

Cristoffel symbols:

Γ_{ik}^t	t	l	θ	φ
t	0	$\frac{\rho^2(r^2g'_{tt}+ff'\sin^2\theta)}{2g}$	$\frac{f^2\rho^2\sin(2\theta)}{2g}$	0
l	$\frac{\rho^2(r^2g'_{tt}+ff'\sin^2\theta)}{2g}$	0	0	$\frac{r^2\rho^4(f/r^2)'}{2g}$
θ	$\frac{f^2\rho^2\sin(2\theta)}{2g}$	0	0	0
φ	0	$\frac{r^2\rho^4(f/r^2)'}{2g}$	0	0

Γ_{ik}^l	t	l	θ	φ
t	$\frac{1}{2}g'_{tt}$	0	0	$\frac{1}{2}f'\sin^2\theta$
l	0	0	0	0
θ	0	0	$-rr'$	0
φ	$\frac{1}{2}f'\sin^2\theta$	0	0	$-rr'\sin^2\theta$

Γ_{ik}^θ	t	l	θ	φ
t	0	0	0	$\frac{f\sin(2\theta)}{2r^2}$
l	0	0	r'/r	0
θ	0	r'/r	0	
φ	$\frac{f\sin(2\theta)}{2r^2}$	0	0	$-\frac{1}{2}\sin(2\theta)$

Γ_{ik}^φ	t	l	θ	φ
t	0	$\frac{\rho^2(fg'_{tt}-f'g_{tt})}{2g}$	$\frac{fr^2g_{tt}\sin(2\theta)}{-2g}$	0
l	$\frac{\rho^2(fg'_{tt}-f'g_{tt})}{2g}$	0	0	$\frac{\rho^2(ff'\sin^2\theta+2g_{tt}rr')}{2g}$
θ	$\frac{fr^2g_{tt}\sin(2\theta)}{-2g}$	0	0	$\frac{r^2\sin(2\theta)(g_{tt}r^2+f^2\sin^2\theta)}{2g}$
φ	0	$\frac{\rho^2(ff'\sin^2\theta+2g_{tt}rr')}{2g}$	$\frac{r^2\sin(2\theta)(g_{tt}r^2+f^2\sin^2\theta)}{2g}$	0

(22)

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